5. Tutorial on Optical Sources and Detectors

June 4th 2012

Problem 1: Spontaneous Emission in an Amplifier

a) Combining the spontaneous and stimulated emission in a fiber amplifier, the power evolves according to the differential equation from section 2.2.2 of the lecture notes:

$$\frac{\mathrm{d}P(z)}{\mathrm{d}z} = (N_2 - N_1)n\sigma(f_s)P(z) + \xi(f_s)$$

For a fiber section of length dz, the spontaneous emission within the filter bandwidth B is given by $\xi(f_s) = N_2 n_{eg} M_T B\sigma(f_s)hf$, where n_{eg} is the effective group refractive index of the fiber and M_{T} the number of transversal modes.

Show that with a given initial power $P(0)$ this differential equation can be solved to:

$$P(z) = P(0)G_s + (G_s - 1)n_{spon}M_T Bhf$$

Where $G_s = \exp(gL)$ describes the single pass gain and the population inversion is defined as $n_{spon} = \frac{N_2}{N_2 - N_1}$

➔ The first order linear ordinary differential equation can be solved by integration:

$$\frac{dP(z)}{dz} = \underbrace{(N_2 - N_1)n\sigma(f_s)}_{g} P(z) + \xi(f_s)$$

$$\frac{\mathrm{d}P(z)/\mathrm{d}z}{gP(z) + \xi(f_s)} = 1$$

$$\int_0^z \frac{\mathrm{d}P(\tilde{z})/\mathrm{d}\tilde{z}}{gP(\tilde{z}) + \xi(f_s)}\mathrm{d}\tilde{z} = \int_0^z 1\mathrm{d}\tilde{z}, \quad \text{substitution: } u = gP(\tilde{z}) + \xi(f_s),\ \mathrm{d}u = gP'(\tilde{z})\mathrm{d}\tilde{z}$$

$$\frac{1}{g}\int_{u(0)}^{u(z)} \frac{1}{u}\mathrm{d}u = \int_0^z 1\cdot \mathrm{d}\tilde{z}$$

$$\frac{1}{g}\left[\ln(u)\right]_{u(0)}^{u(z)} = [\tilde{z}]_0^z, \quad \text{resubstitution: } \frac{1}{g}\left[\ln\left(gP(\tilde{z}) + \xi(f_s)\right)\right]_0^z = [\tilde{z}]_0^z$$

$$\left[\ln\left(gP(z) + \xi(f_s)\right) - \ln\left(gP(0) + \xi(f_s)\right)\right] = gz$$

$$\ln\left(\frac{gP(z) + \xi(f_s)}{gP(0) + \xi(f_s)}\right) = gz$$

$$gP(z) + \xi(f_s) = e^{gz}\left(gP(0) + \xi(f_s)\right)$$

$$P(z) = \frac{e^{gz}}{g}\left(gP(0) + \xi(f_s)\right) - \frac{\xi(f_s)}{g}$$

with $\mathrm{G}_s = e^{gz}$, $g = (\mathrm{N}_1 - N_2)n_{eg}\sigma(f_s)$ and $\xi(f_s) = N_2 n_{eg} M_T B\sigma(f_s)hf$

$$P(z) = P(0)G_s + \frac{G_s - 1}{(N_2 - N_1)n_{eg}\sigma(f_s)} N_2 n_{eg} M_T B\sigma(f_s)hf$$

$$P(z) = P(0)G_s + (G_s - 1)n_{spon}M_T Bhf$$

b) An unsaturated laser amplifier of length d and gain coefficient $g(f)$ amplifies an input signal $P_{\mathrm{s}}(0)$ of the frequency f and introduces amplified spontaneous emission (ASE) at a rate ξ (per unit length). The amplified signal power is $P_s(d)$ and the ASE at the output is $P_{ASE}(d)$. Sketch the dependence of the ratio $P_s(d)/P_{ASE}(d)$ on the product of the amplifier gain coefficient and length: $g(f)d$.

➔ From a) we find:

$$\frac{P_s(d)}{P_{ASE}(d)} = \frac{P(0)G_s}{(G_s-1)n_{spon}M_T Bhf} = \frac{P(0)e^{gd}}{\left(e^{gd}-1\right)n_{spon}M_T Bhf}$$

For large $gd, e^{gd} >> 1 \rightarrow (G_s-1) \approx G_s$ and $\frac{P_s(d)}{P_{ASE}(d)} \approx \frac{P(0)}{n_{spon}M_T Bhf} = \text{const}$

For $gd \rightarrow 0, \frac{P_s(d)}{P_{ASE}} = \lim\limits_{G_s \rightarrow 1} \frac{P(0)}{(G_s-1)n_{spon}M_T Bhf} = \infty$

Problem 2: Condition for optical gain in a semiconductor

It has been shown in the lecture that the photon emission rate is approximately the difference between the rate of absorption and stimulated emission:

$$\frac{dN_p}{dt} \approx r_{st} - r_{ab} = N_p A_{12}\left(n(W_2)\,p(W_1) - p(W_2)\,n(W_1)\right),$$

where n (p) stands for the number of electrons (holes) at the energy level W_1 or W_2 respectively and W_1 is an energy level within the valence band whereas W_2 denotes an energy level within the conduction band. For net optical gain the photon emission rate needs to be positive.

a) In the case of a semiconductor in thermal equilibrium show that this condition cannot be fulfilled!

➔ In order to get net gain, i.e. photons are created, the following condition would need to be fulfilled:

$$\frac{dN_p}{dt} > 0 \quad \Rightarrow \quad n(W_2)\,p(W_1) > p(W_2)\,n(W_1)$$

with

$$n(W_2) = \rho_C(W_2)\cdot f(W_2)$$
$$p(W_1) = \rho_V(W_1)\cdot\left[1 - f(W_1)\right]$$
$$p(W_2) = \rho_C(W_2)\cdot\left[1 - f(W_2)\right]$$
$$n(W_1) = \rho_V(W_1)\cdot f(W_1)$$

This leads the following inequality:

$$f(W_2) \overset{!}{>} f(W_1).$$

However, since

$$f(W) = \frac{1}{1+\exp\left\{\dfrac{W-W_F}{kT}\right\}}$$

and $W_2 > W_1$ (W_2 is within the conduction band and W_1 is within the valence band) $f(W_2) < f(W_1)$.

b) Now consider a semiconductor that has been displaced from thermal equilibrium. However this perturbation is slow enough for the semiconductor to reach a quasi thermal equilibrium with the two quasi Fermi levels W_{Fn} and W_{Fp}. Show that it is now possible to obtain net optical gain! Derive the condition that the quasi Fermi levels need to fulfill in order to obtain net optical gain!

➔ Now we look at the same equation as above however this time we consider two separate Fermi functions for the conduction and the valence band with the respective quasi Fermi levels W_{Fn} and W_{Fp}

$$\begin{aligned} n(W_2) &= \rho_C(W_2)\cdot f_n(W_2) \\ p(W_1) &= \rho_V(W_1)\cdot\left[1-f_p(W_1)\right] \\ p(W_2) &= \rho_C(W_2)\cdot\left[1-f_n(W_2)\right] \\ n(W_1) &= \rho_V(W_1)\cdot f_p(W_1) \end{aligned}$$

leading to:

$$f_n(W_2) \overset{!}{>} f_p(W_1)$$

inserting now the respective Fermi functions

$$f_n(W) = \frac{1}{1+\exp\left\{\dfrac{W-W_{Fn}}{kT}\right\}} \quad \text{and} \quad f_p(W) = \frac{1}{1+\exp\left\{\dfrac{W-W_{Fp}}{kT}\right\}}$$

one obtains that: $W_2 - W_1 < W_{Fn} - W_{Fp}$! Since the difference of W_1 and W_2 corresponds to the energy of an emitted photon, which in turn has to be larger than the bandgap one can conclude that the quasi Fermi levels need to be separated by more than the bandgap energy. This in turn means that at least one quasi Fermi level needs to be within the respective (conduction or valence) band.

c) How can population inversion in a semiconductor be achieved? Name at least two different ways!

➔ Optical pumping. However one needs to consider that optical pumping is only possible in a more than 2 level system.
Electrical pumping of a pn junction

Questions and Comments:

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