2. Tutorial on Optical Sources and Detectors

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Problem 1: Density of modes in an optical resonator

Consider a three-dimensional optical resonator constructed of three pairs of parallel mirrors that form the walls of a box with edge lengths L_x, L_y, and L_z. Eigenmodes of the resonator have to fulfill the boundary conditions at the sidewalls and are represented by standing-wave solutions with discrete components k_x, k_y, and k_z of the wavevector. In the following, we will calculate the density of these modes per unit of volume and frequency. The derivation can be performed in analogy to the density of states of electrons in a semiconductor.

a) Find the condition for k_x, k_y, and k_z that need to be fulfilled in order to obtain standing waves within the three-dimensional resonator. For simplicity, assume that the resonator is filled with air of unity refractive index.

➔ Standing wave in the resonator $L = \nu \frac{\lambda}{2} \quad \Rightarrow \lambda = \frac{2L}{\nu}$

$$k_x = \frac{2\pi}{\lambda} = \nu_x \frac{2\pi}{2L_x} = \nu_x \frac{\pi}{L_x}, \quad k_y = \nu_y \frac{\pi}{L_y}, \quad k_z = \nu_z \frac{\pi}{L_z}, \quad \nu_{x,y,z} \in \mathbb{N}$$

b) Each pair of modes (considering two polarizations) is represented by a positive triple of k_x, k_y, and k_z, all of which are fulfilling the boundary condition. Calculate the volume V_k that a single optical mode occupies in k-space?

$$\Delta V_k = \frac{1}{2} k_x k_y k_z = \frac{1}{2} \frac{\pi^3}{L_x L_y L_z} = \frac{\pi^3}{2V}$$

c) Calculate the number of optical modes $M(f)$ within the frequency interval 0 and f. In k-space these modes lie within the positive octant of a sphere, the radius of which is related to f. Calculate the density of modes, i.e. the number of optical modes per volume and per frequency interval:

$$\rho(f) = \frac{1}{V} \frac{dM(f)}{df}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi n f}{c},$$

$$V_k(f) = \frac{1}{8}\frac{4}{3}\pi r^3 = \frac{1}{6}\pi\left(\frac{2\pi n f}{c}\right)^3, \quad 1/8 \text{ sphere with radius } k \text{ is occupied by all modes in k-space}$$

➔

$$M(f) = \frac{V_k(f)}{\Delta V_k} = \frac{1}{6}\pi\left(\frac{2\pi n f}{c}\right)^3 \cdot \frac{2V}{\pi^3} = \frac{8\pi}{3}\frac{n^3 f^3 V}{c^3}, \quad \text{volume of all modes devided by volume of a single mode}$$

$$\rho(f) = \frac{1}{V}\frac{\mathrm{d}M}{\mathrm{d}f} = \frac{8\pi n^3 f^2}{c^3} \quad \text{differentiated with respect to } f$$

d) In Section 2.1.2 of the lecture notes the following quantity is denoted as the "average number of photons per mode":

$$\bar{N}_p = \frac{\lambda_0^3}{8\pi h n^3} u(f_0)\,.$$

Show that this formula can be derived by dividing the number of photons N_p in a resonator by the number of modes in the resonator. Is the average number of photons per mode dependent on the parameters of the model resonator?

➔ The number of available modes within a certain frequency interval [f; f + df] is given by the product $N_{modes} = \rho(f)\cdot V_{res}\cdot \mathrm{d}f$.
With frequency in [f; f + df] inside the cavity and with $u(f)$ the spectral energy density, the number of photons follows as $N_{phot} = V\frac{u(f)}{hf}\mathrm{d}f$

$$\bar{N}_p = \frac{N_{phot}}{N_{modes}} = \frac{u(f)V_{res}df}{hf\rho(f)V_{res}df} = \frac{c^3 u(f)}{hf\,8\pi n^3 f^2} = \frac{\lambda^3}{8\pi h n^3}u(f)$$

Problem 2: Carrier Concentration in Semiconductors

The densities n of conduction band (CB) electrons and the density p of valence band (VB) holes of an undoped semiconductor can be calculated with the help of the density of states $\rho_C(W)$ of the CB and $\rho_V(W)$ of the VB, and the Fermi-Dirac distributions $f(W)$ and $[1-f(W)]$ for electrons and holes, respectively:

$$n = \int_{W_C}^{\infty} \rho_C(W)\, f(W)\,\mathrm{d}W$$

$$p = \int_{-\infty}^{W_V} \rho_V(W)\left[1-f(W)\right]\mathrm{d}W$$

.

If the energetic distance of the Fermi level from the band edges is $|W_{C,V} - W_F| > 3kT$, then the Fermi functions can be simplified by the Boltzmann approximation

$$f(W) = \exp\left\{-\frac{W-W_F}{kT}\right\}, \quad \text{for } (W-W_F) > 3kT,$$

$$f(W) = 1-\exp\left\{-\frac{W_F-W}{kT}\right\}, \quad \text{for } (W-W_F) < -3kT,$$

which shall be used for the calculations below. In practice, however, this condition is usually not fulfilled, e.g. for a highly doped laser diode, the Fermi level is very close to or even inside the band. Thus the following considerations show the principle only.

a) Show that the maximum of the electron distribution $\rho_C(W)\,f(W)$ is found at the energy $kT/2$ above the band edge W_C of the CB.

➔ In a undoped bulk semiconductor the density of states is $\rho_c(W) = \frac{4\pi(2m_n)^{3/2}}{h^3}\sqrt{W-W_c}$ and the Fermi distribution of the holes with the Bolzmann approximation: $f(W) \cong \exp\left(-\frac{W-W_F}{kT}\right)$

$$\frac{\mathrm{d}\rho_c(W)f(W)}{\mathrm{d}W}=0$$

$$\frac{4\pi(2m_n)^{(3/2)}}{h^3}\exp\left(-\frac{W-W_F}{kT}\right)\left(\frac{1}{2\sqrt{W-W_c}}-\frac{\sqrt{W-W_c}}{kT}\right)=0$$

$$\left(\frac{1}{2\sqrt{W-W_c}}-\frac{\sqrt{W-W_c}}{kT}\right)=0$$

$$\frac{1}{2\sqrt{W-W_c}}=\frac{\sqrt{W-W_c}}{kT}$$

$$W_c+\frac{1}{2}kT=W$$

b) Solve the above given integrals for n and p by using the Boltzmann approximation and write the carrier concentration as

$$n=N_C\exp\left(-\frac{W_C-W_\mathrm{F}}{kT}\right)$$
$$p=N_V\exp\left(-\frac{W_\mathrm{F}-W_V}{kT}\right),$$

where N_C and N_V are the effective density of states of the CB and VB, respectively.

➔ The carrier concentration n can be written as:

$$n=\int_{W_C}^{\infty}\rho_C(W)\,f(W)\,\mathrm{d}W$$
$$=\int_{W_C}^{\infty}\frac{4\pi(2m_n)^{3/2}}{h^3}\sqrt{W-W_c}\cdot\exp\left(-\frac{W-W_F}{kT}\right)\mathrm{d}W$$

Expanding the integral with $1=\frac{\sqrt{kT}}{\sqrt{kT}}\exp\left(\frac{W_c-W_c}{kT}\right)$

$$n=\frac{4\pi(2m_n)^{3/2}}{h^3}\sqrt{kT}e^{-\frac{W_c-W_F}{kT}}\int_{W_c}^{\infty}\sqrt{\frac{W-W_C}{kT}}\cdot e^{-\frac{W-W_c}{kT}}\mathrm{d}W$$

To solve the integral use the substitution:

$$u=\sqrt{\frac{W-W_C}{kT}}\quad\Rightarrow\quad dW=2kTu\cdot du$$
$$W=W_C\quad\Rightarrow\quad u=0$$
$$W=\infty\quad\Rightarrow\quad u=\infty$$

$$n=2\frac{4\pi\left(2m_nkT\right)^{3/2}}{h^3}\cdot e^{-\frac{W_C-W_F}{kT}}\underbrace{\int_0^{\infty}u^2\cdot e^{-u^2}du}_{=\frac{\sqrt{\pi}}{4}}$$

$$n=\underbrace{2\left(\frac{2\pi m_nkT}{h^2}\right)^{3/2}}_{N_C}\cdot e^{-\frac{W_C-W_F}{kT}}$$

The calculation of p can be done analog.

$$p = \underbrace{2\left(\frac{2\pi m_p kT}{h^2}\right)^{3/2}}_{N_V} \cdot e^{-\frac{W_F - W_V}{kT}}$$

c) Take the results from b) and determine the energetic distance of the Fermi level relative to the band edges.

➔ In the intrinsic case it follows for the carrier densities $n = p$.

$$N_C \exp\left(-\frac{W_C - W_F}{kT}\right) = N_V \exp\left(-\frac{W_F - W_V}{kT}\right)$$

Solving for W_F:

$$W_F = \frac{W_C + W_V}{2} + \frac{1}{2} kT \ln\frac{N_V}{N_C} = \frac{W_C + W_V}{2} + \frac{3}{4} kT \ln\frac{m_p}{m_n}$$

d) The bandgap energy of GaAs is W_G = 1.424 eV, the effective masses are $m_n = 0.067 m_0$ and $m_p = 0.48 m_0$, where m_0 is the electron rest mass. Determine the intrinsic carrier concentrations n and p of GaAs at T =300 K by using the results of b).

$$np = n_i^2 = N_C N_V e^{-\frac{W_G}{kT}}$$

➔
$$n_i^2 = 4.345 \cdot 10^{24}\,\mathrm{m}^{-6}$$
$$n = p = 2.08 \cdot 10^{12}\,\mathrm{m}^{-3} = 2.08 \cdot 10^{6}\,\mathrm{cm}^{-3}$$

Problem 3: Fiber optic communication system

Consider a fiber optic communication system that transmits data with a data rate of 40 Gbit/s over a distance of 100 km. The data is modulated onto a carrier at 1550 nm and the fiber has an average attenuation (considering splices etc.) of 0.3 dB/km. The transmitter couples an average power of 2 mW into this fiber.

Assume that the '0' bits don't carry any power whereas the '1' bits have a constant power throughout the whole bit slot (non-return-to-zero (NRZ) modulation). Further suppose that the probability for transmitting a '1' is equal to the probability of transmitting a '0'.

How many photons arrive at the receiver during a single '1' bit?

➔ The Attenuation in the system is 0.3 dB/km · 100 km = 30 dB
at 2mW input we get 3dBm – 30dB = -27 dBm= 2 µW
‘1’ and ‘0’ are distributed equally, but only the ‘1’ carries power which equals to 4 µW of average power at the receiver if only ‘1’s were transmitted
Average number of photons $\bar{N}$ during a ‘1’ calculates as:

$$Power = \frac{Energy}{Time} \quad \Rightarrow P_{av,1} = \frac{\bar{N} \cdot hf}{T_{bitslot}}$$

$$\bar{N} = \frac{P_{av,1} T_{bitslot}}{hf} = \frac{P_{av,1} T_{bitslot} \lambda}{hc} = \frac{4\mu W (40 \cdot 10^9 bit/s)^{-1} 1550nm}{hc} \approx 780$$

Questions and Comments:

Matthias Lauermann
Building: 30.10, Room: 2.32
Phone: 0721/608-41695
Email: Matthias.Lauermann@kit.edu

Jörg Pfeifle
Building: 30.10, Room: 2.23
Phone: 0721/608-48954
Email: Joerg.Pfeifle@kit.edu